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NOTE

Beware of Maxwell's Divergence Equations

Using the constitutive relations $\mathbf{D} = \varepsilon\mathbf{E}$, $\mathbf{B} = \mu\mathbf{H}$, the Maxwell equations

$$\begin{aligned} &\nabla \wedge \mathbf{E} + \partial_t(\mu\mathbf{H}) = 0 \quad (a), && \nabla \cdot \mu\mathbf{H} = 0 \quad (b) \\ &\nabla \wedge \mathbf{H} - \partial_t(\varepsilon\mathbf{E}) = \mathbf{j} \quad (c), && \nabla \cdot \varepsilon\mathbf{E} = \rho \quad (d) \end{aligned} \tag{1}$$

become a system of eight equations for only six unknown. An algebraic system of equations in this case would be overdetermined. And in fact, it is generally inferred that the divergence equations are redundant [4] since (1)(b) is a consequence of (1)(a) provided that $\nabla \cdot \mu\mathbf{H} = 0$ at some time t_0 and since (1)(d) is a consequence of (1)(c) taking into account the condition of continuity $\nabla \cdot \mathbf{j} + \partial_t\rho = 0$. For harmonic fields let us quote Jones [2], "Therefore, for harmonic fields, which are nonstatic, it is sufficient to employ (2.9) and (2.10) [our Eqs. (1)(a), (c), with ∂ replaced by $i\omega$] plus the harmonic form of the equation of continuity."

Recently, in a very important work, Jiang *et al.* [1] proved that the divergence equations are not redundant and that neglecting these equations is at the origin of spurious solutions in computational electromagnetics. In this note we give (without proof) a simple but mathematically demanding result obtained by Muller [3] many years ago and illustrating the role of the divergence equations.

We start with the harmonic Maxwell equations, ω, ε, μ, real positive

$$\begin{aligned} &\nabla \wedge \mathbf{H} + i\omega\varepsilon\mathbf{E} = 0, && \nabla \cdot \mathbf{H} = 0 \\ &\nabla \wedge \mathbf{E} - i\omega\mu\mathbf{H} = 0, && \nabla \cdot \mathbf{E} = 0 \end{aligned} \tag{2}$$

supplying the partial differential equation for $\mathbf{E}$, Δ the laplacian operator, $k^2 = \omega^2\varepsilon\mu$,

$$\Delta\mathbf{E} + k^2\mathbf{E} = 0 \quad (a), \qquad \nabla \cdot \mathbf{E} = 0 \quad (b). \tag{3}$$

We use the notations (the vector $\mathbf{r}_0$ is a function only of the direction)

$$\mathbf{r} = r\,\mathbf{r}_0, \qquad {\mathbf{r}_0}^2 = 1, \qquad r \geq 0. \tag{4}$$

Then, Muller [3] proved that to satisfy the divergence equation (3)(b) a solution of the Helmholtz equation (3)(a) must fulfill the asymptotic condition

$$r(\mathbf{E}(r\,\mathbf{r}_0)) = o(1), \qquad r \Rightarrow \infty. \tag{5}$$

The notation $o(1)$ means that $|r\,\mathbf{E}(r\,\mathbf{r}_0)|$ behaves as a constant for $r \Rightarrow \infty$.

To prove the existence of such a solution for the Maxwell equations (2), Muller considered on the unit sphere Ω a vector field $\mathbf{F}(\mathbf{r}_0)$ which has only tangential components $\mathbf{r}_0 \cdot \mathbf{F}(\mathbf{r}_0) = 0$ and such as each of its cartesian components satisfies on Ω the relation

$$\lim_{r\Rightarrow\infty}(1/kr \log \int_\Omega |\mathbf{F}(r\,\mathbf{r}_0)|^2\, d\omega) = R \leq \infty. \tag{6}$$

Then, there exists for $|\mathbf{r}| \geq R$ a solution $\mathbf{E}(r\,\mathbf{r}_0)$ of (3)(a) such that

$$\mathbf{E}(r\,\mathbf{r}_0) = e^{ikr}/r\,\mathbf{F}(\mathbf{r}_0) + o(1/r) \tag{7}$$

so that the condition (5) is fulfilled and this solution satisfies the divergence equation (3)(b).

From Maxwell's equations we get

$$\mathbf{H}(\mathbf{r}) = -i/\omega\mu\, \nabla \wedge \mathbf{E}(\mathbf{r}) \tag{8}$$

and using (7), a simple calculation shows that

$$\mathbf{H}(r\,\mathbf{r}_0) = k/\omega\mu e^{ikr}/r(\mathbf{r}_0 \wedge \mathbf{E}(r\,\mathbf{r}_0) + o(1/r)) \tag{9}$$

so that $\mathbf{H}(\mathbf{r})$ is a solution of the equations $\Delta\mathbf{H} + k^2\mathbf{H} = 0$, $\nabla \cdot \mathbf{H} = 0$. In addition these solutions satisfy the radiation conditions

$$\begin{aligned} &\omega\varepsilon(\mathbf{r}_0 \wedge \mathbf{E}) - k\mathbf{H} = o(1/r), && \mathbf{E} = o(1/r) \\ &\omega\mu(\mathbf{r}_0 \wedge \mathbf{H}) + k\mathbf{E} = o(1/r), && \mathbf{H} = o(1/r). \end{aligned} \tag{10}$$

This result for the system of equations (2), the simplest that one can imagine in electromagnetism, shows that one has to work with the divergence equations. As noticed in [5] although the components of $\mathbf{E}$ and $\mathbf{H}$ are solutions of the wave equation, "choosing the components to satisfy

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$\nabla \cdot \mathbf{E} = 0$, $\nabla \cdot \mathbf{H} = 0$ couples the components, as do the boundary and initial conditions so the solution to problems is again more than just the solution of the scalar wave equation.''

A thorough discussion on the role of the divergence equations in computational electromagnetics is given in [1].

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